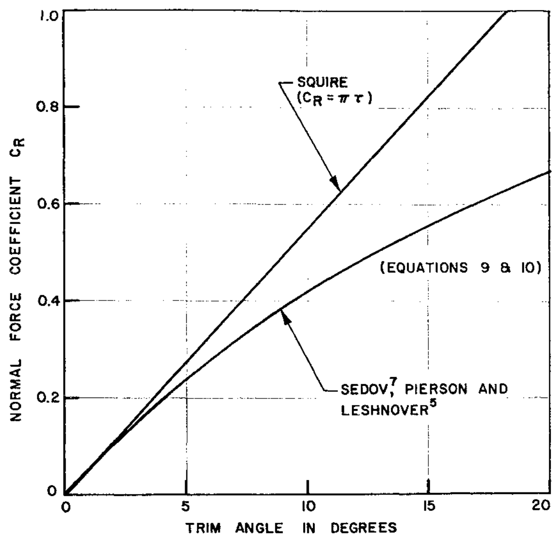

Fig. 2 Normal force coefficient for flat plate, as function of trim angle, in limit $F_l \to \infty$.

$(F_l \to \infty)$ of

$$C_R = \frac{2\pi \sin\tau}{1+\cos\tau - (1-\cos\tau)\log\left(\dfrac{1-\cos\tau}{2\cos\tau}\right) + \pi\sin\tau} \tag{9}$$

This falls significantly below the Squire value of $C_R = \pi\tau$ above trim angles of 4-5 deg, as shown in Fig. 2. If Eq. (9) is substituted into Eq. (4), Pierson and Leshnover's result is reproduced, however.

Sedov[7] obtained the same result as Eq. (9), expressed as

$$C_R = \frac{2\pi}{\cos\dfrac{\tau}{2} + \pi + \tan\dfrac{\tau}{2}\log\left(\cot^2\dfrac{\tau}{2} - 1\right)} \tag{10}$$

We conclude that Squire's analysis applies only to small trim angles—as he states in his assumptions—and that a better approximation might be Eq. (9) or (10) divided by Squire's factor $[1+(5/8)(\pi/F_l^2)]$ as an empirical correction for finite fluid weight.

When the plate is not flat the trim angle τ can be measured from zero lift trim, but the total resistance is now $R\beta$ rather than $R\tau$, β being the induced drag angle. Thus Eq. (3) becomes

$$D_w + D_j = R\sin\beta \tag{11}$$

leading to

$$\frac{\delta}{l} = \frac{C_R}{2(1+\cos\theta)}\left[\sin\beta - \frac{C_R\cos^2\beta}{2F_l^2}\right] \tag{12}$$

where θ is the jet angle above the horizontal.

Since methods exist for calculating the two-dimensional pressure distribution on a cambered plate[6] (Squire's[1] methodology is readily adapted, and Sedov[7] gives results by M.I. Gurevich for a circular arc profile, using Sedov's methodology), it is possible to find the induced drag angle β and hence deduce δ/l from Eq. (11).

Any pressure field (such as that generated in the bubble of a surface effect ship) which does not have local regions of pressure equal to $\frac{1}{2}\rho u_0^2$, will not develop a jet. The same would be true (under the same conditions) of a cambered planing plate which had the same shape as the water surface in a pressure field.

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H80-015

Wave Focusing and Hydraulic Jump Formation

20006

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Introduction

THE analysis of hydraulic jumps in shallow water usually centers on jump conditions obtained by relating the upstream and downstream solutions for mean height and speed through mass and momentum conservation. In this classic approach, a resultant decrease of energy across the discontinuity is assumed to occur through turbulent dissipation (e.g., see Lamb[1]). Recently, this author,[2] expanding on some work of Benjamin and Lighthill,[3] generalized the foregoing treatment to allow for the addition of finite amplitude waves. Four quantities, namely, the mean height h, the mean speed U, the wave energy density E, and the wave number k now appear as unknowns; upstream and downstream conjugate solutions were obtained by conserving total mass, total momentum, total energy, and wave crest number. The results of the author's study showed how, in a nondissipative system, the usual loss of mean flow energy could be attributed to the appearance of enhanced downstream radiating waves characterized by significantly increased energy density and wave number.

The exact role played by these waves in setting up the discontinuous flow, however, is unclear; certainly, these "unstable" waves, which experience phenomenal growth through the shock layer, must play some dominant part in a very interesting dynamical process which unfortunately cannot be uncovered by studying jump conditions alone. This Note provides an exploratory study on hydraulic jump for-

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mation by examining the nonlinear mean flow response to these "unstable" finite amplitude waves in inviscid nonuniform media. An early study by Kantrowitz[4] first showed how wave focusing and trapping in slowly varying convergent/divergent gasdynamic nozzles resulted in large flow gradients and shock formation. It therefore seems plausible, especially in view of a well-known analogy connecting shallow-water waves with one-dimensional compressible flow, that the same physical mechanism would apply. That this appears to be the case is suggested in the following analysis; the end result of this breakdown mechanism is, presumably, a discontinuous flow characterized by the jump conditions of Ref. 2, by analogy to the gasdynamic example of Kantrowitz.

Analysis

Hydraulic jump formation in its fullest detail involves unsteady nonlinear interactions in nonuniform viscid media. However, some insight is gained by examining the steady propagation of inviscid gravity waves over a shallow sloping bottom. The required modulation equations for slowly varying wavetrains are readily available (e.g., see Whitham[5] or Phillips[6]); the shallow water equations for total energy and total mass conservation integrate, respectively, to

$$\rho hU(\tfrac{1}{2}U^2+gh)+U\tilde{S}+(U+\tilde{C})E=\alpha \tag{1}$$

$$\rho hU=\beta \tag{2}$$

where we have adopted Whitham's nomenclature. Here, α and β are positive fluxes, ρ is the fluid density, g is the acceleration due to gravity, U is the "mass flow velocity" (accounting for wave-induced changes to the unperturbed mean flow), $\tilde{C}=(gh)^{1/2}$ and $\tilde{S}=3E/2$. Two additional equations, namely, those for wave crest and momentum conservation, are required to fix the unknowns h, U, E, and k; the sloping bottom ordinate $d(x)$, where x is the streamwise coordinate, is the only input required to determine the entire flowfield completely (this variable $d(x)$ adds momentum to the flow). We will now assume that the solution for any arbitrary $d(x)$ is known and examine some properties of the governing modulation equations leading to hydraulic behavior.

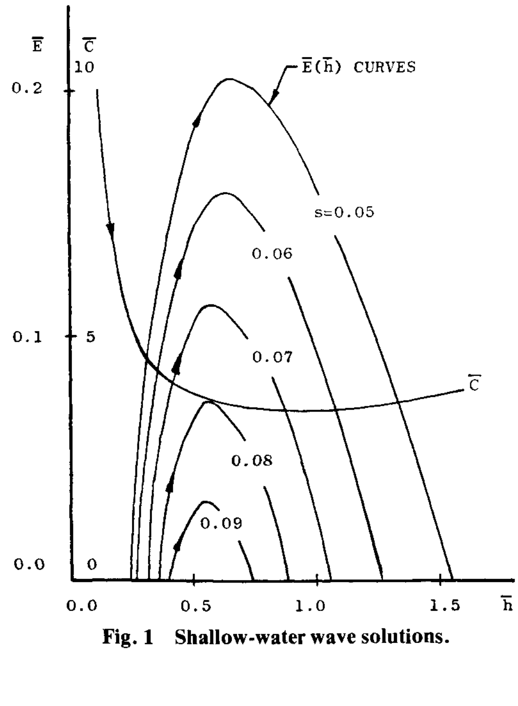

Fig. 1 Shallow-water wave solutions.

First introduce the nondimensional variables $\bar{h}$, $\bar{U}$, $\bar{E}$, and $\bar{C}$ defined by

$$h=h_0\bar{h}=\left(\frac{5\beta}{2\rho g^{1/2}}\right)^{2/3}\bar{h} \tag{3}$$

$$E=\alpha\bar{E}/(gh_0)^{1/2} \tag{4}$$

$$U=\left(\frac{4}{25}\frac{g\beta}{\rho}\right)^{1/3}\bar{U} \tag{5}$$

$$C=\left(\frac{4}{25}\frac{g\beta}{\rho}\right)^{1/3}\bar{C} \tag{6}$$

noting that C, the group velocity, equals $U+(gh)^{1/2}$. The nondimensional parameter

$$s=\left(\frac{2\beta^5 g^2}{625\rho^2\alpha^3}\right)^{1/3}>0 \tag{7}$$

also enters into the analysis. These definitions result in the two following normalized formulas,

$$\bar{E}=\frac{1-\dfrac{s}{\bar{h}^2}-\dfrac{25}{2}s\bar{h}}{\dfrac{1}{\bar{h}}+\bar{h}^{1/2}} \tag{8}$$

$$\bar{C}=\frac{1}{\bar{h}}+\frac{5}{2}\bar{h}^{1/2} \tag{9}$$

The group velocity $\bar{C}$ and the wave energy density $\bar{E}$ may be plotted against x, of course, once $d(x)$ is given; however, as in Fig. 1, it is more instructive to plot $\bar{E}$ and $\bar{C}$ against $\bar{h}(x)$ itself.

Now imagine a wave of fixed s originating in a region of small $\bar{h}$. As the wave progresses into regions of higher $\bar{h}$ (following trajectories defined by the directional arrows of Fig. 1) the wave energy density increases and the group velocity decreases. At a point near the minimum in $\bar{C}$, small increases in $\bar{E}$ are complemented by large increases in $\bar{h}$, these results being exemplified by a relative maximum appearing in the $\bar{E}(\bar{h})$ vs $\bar{h}$ plots. Because $\bar{U}$ varies inversely with $\bar{h}$, increases in $\bar{E}$ are also accompanied by sudden decreases in mean flow speed; also, since the real frequency $\omega=Ck$ is a constant of the flow, the wave number k increases. (Figure 1 shows the group velocity decreasing as $\bar{h}$ increases.) These results agree qualitatively with Ref. 2, as they should, but the identification of this jump formation with a region of small $\bar{C}$, that is, wave focusing, is important. Figure 1 provides a new interpretation of Eqs. (1) and (2); it furnishes a tentative clue connecting hydraulic jump formation in shallow water to shock formation in one-dimensional Laval nozzles. Further propagation into zones near maximum allowable $\bar{h}$ are accompanied by smooth flow and small wave energy densities.

In the present problem the wave focusing is accompanied by a required wave amplification necessary to insure energy absorption from the mean flow. The modulation equations are globally nondissipative, again, but the wave growth here arises from the action of radiative stresses. The steady-wave energy equation governing this flow takes the form[5]

$$\frac{\partial}{\partial x}CE=-\frac{3}{2}U'(x)E \tag{10}$$

where C is the group velocity. Since the wave under consideration propagates into regions of decreasing $U(x)$, the derivative $U'(x)$ is negative, thus guaranteeing energy growth for all wave packets regardless of wave number. This energy absorption is responsible for the loss of mean flow energy; Fig. 1, of course, indicates the rapidity through which the mean flow responds nonlinearly.

Conclusions

The wave growth considered in this Note arises from the effects of radiation stress and from the energy accumulation due to ray coalescence; that is, from Eq. (10),

$$E(x)=\frac{(EC)_{\mathrm{ref}}}{C(x)}\exp\left\{-\frac{3}{2}\int_{x_{\mathrm{ref}}}^{x}\frac{U'(\xi)\,\mathrm{d}\xi}{C(\xi)}\right\} \tag{11}$$

Equation (11) is well known, of course, but it is the connection between large $\bar{E}$'s and large $\partial\bar{h}/\partial\bar{E}$ observed near the minimum of $\bar{C}$ that elicits the greatest interest. The $1/C$ effect, apparently, successfully forces a sudden mean flow response manifested through the appearance of hydraulic jumps that would not "normally" arise for large C. A variant of this argument, interestingly, explains shock formation in Laval nozzles.[4] The relationship between the present work and Landahl's[7] theory for wave breakdown merits some discussion. Landahl's linear theory for wave breakdown in continuous media identifies "strong instability" with zeros in the wave group velocity; this ray coalescence results in a local accumulation in energy density which, if amplified by a net positive contribution arising from radiation stress and wave dissipation, would elicit a sudden nonlinear mean flow response and, thus, hydraulic behavior. For the inviscid bores considered here, Landahl's explanation appears to be the correct one; nonlinear theory, of course, replaces group velocity zeros by relative minimums, so that passage through the focus is possible. Waves in real flows, as may be the present case, are slightly nonconservative; they admit complex frequencies obscuring, somewhat, the role of the "real" group velocity. This point, highlighted by a number of authors in different contexts, is addressed in a recently published high-order study clarifying the conditions under which Landahl's modifications for weak dissipation hold.[8]

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